DOI: 10.2478/auom-2018-0042
An. Şt. Univ. Ovidius Constanţa **Vol. 26(3),2018, 213–227**

Some Topological and Algebraic Properties of α-level Subsets' Topology of a Fuzzy Subset

B. O. Onasanya and S. Hoskova-Mayerova

Abstract

The theory of fuzzy sets, since its foundation, has advanced in a wide range of means and in many fields. One of the areas to which fuzzy set theory has been applied extensively is mathematical programming. Nevertheless, the applications of fuzzy theory can be found in e.g. logic, decision theory, artificial intelligence, computer science, control engineering, expert systems, management science, operations research, robotics, and others. Theoretical improvements have been made in many directions. Nowadays it has a lot of applications also on possibility theory, actuarial credibility theory, fuzzy logic and approximate reasoning, fuzzy control, fuzzy data analysis, fuzzy set models in operations research, etc. The aim of this paper is to investigate some topological properties of a set X when the topology defined on it is the collection of all the α-level subsets of a fuzzy subset A of X.

We have been able to establish some results regarding fuzzy cluster level subsets, convergence of level subsets and quasicompactness among others.

1 Introduction

In [28], Zadeh introduced the general theory of fuzzy sets. He laid the foundation for the concept of complement, union, intersection and emptiness of fuzzy set. Chang [5] was the first person to extend these concepts to a topological space. He defined a fuzzy topology on a set X as the collection of all its fuzzy

Key Words: quasicompact, cluster level set, minimal level set.
2010 Mathematics Subject Classification: Primary 54A40; Secondary 03E72.
Received: 8.01.2018
Accepted: 31.01.2018

subsets satisfying three axioms similar to those of a classical fuzzy topological space.

Wong [25] considered some properties of Chang's fuzzy topological space such as cover, sequential compactness and semi compactness and concluded that the theory of fuzzy topological spaces in these regards are less useful. He also, in [26], developed a product and quotient fuzzy topology on Chang's fuzzy topological space. He examined some properties of fuzzy homeomorphism which he regarded as an F-continuous one-to-one mapping between two fuzzy topological spaces such that the inverse also is F-continuous. He further, in [27], introduced the concept of fuzzy point in order to formulate the idea of fuzzy convergence.

However, noting that, according to Chang's fuzzy topological space, constant maps between fuzzy topological spaces are not continuous, Lowen [15] modified the definition of fuzzy topology on X by Chang. Hence, he referred to Chang's topology as a quasi topology. Along side, he formulated and extended some ideas such as continuity and compactness. But Gantner [11] faulted the work of Lowen [15] as losing the concept of generalization which fuzzy topology on X is to yield. Sarka [22] considered some Hausdorff separation axioms by taking the fuzzy elements into consideration and, with this, he hoped to improve on the work of Gantner [11] which he considered to have done a similar thing but for only the crisp points and crisp subsets of fuzzy spaces.

Shostak [23] made a modification that is completely different from the existing concept of fuzzy topology by introducing that a fuzzy set can be open or closed to a degree. Hence a fuzzy topological function

$$\tau(\alpha)=\begin{cases} 1, & \text{if } \alpha \text{ is completely open} \\ \beta, & \text{if } \alpha \text{ is open to a degree } \beta\in(0,1) \\ 0, & \text{if } \alpha \text{ is completely not open.} \end{cases}$$

Chakrabarty *et al*[4] took a completely different line of thought by introducing fuzzy topology on a fuzzy set rather than on X, using a tolerance relation $\overline{S}$. This was considered tolerance topology and the pair $(\overline{A},\overline{S})$ a tolerance space, where $\overline{A}$ is a fuzzy set. Chaudhuri *et al* [6] built on this topology to develop the concepts such as Hausdorffness, regularity, normality and completeness of normality. Das [10] continued in this setting and introduced a product topology and fuzzy topological group.

However, there was a resurgence of gradation of openness of Shostak [23]. This, according to Gregori *et al* [12], makes it easier to avoid the concept of fuzzy point. The concepts of α-level openness, α-interior and α-neighbourhood were also introduced. This idea was also followed by Benchalli [2] to obtain some fuzzy topological properties such as α-Hausdorffness, α-connectedness and α-compactness. Onasanya in [20] also introduced and studied some properties such as fuzzy accumulation (or cluster) points of an α-level subset of a fuzzy topological space instead of that of fuzzy cluster set introduced by Chang in [5]. This also makes it easier to avoid fuzzy points.

Another ideas concerning fuzzy topological hypergroups can be found e.g. in [1, 9, 7, 14, 19] and many others.

We now introduce a setting where the topology we introduce is on the fuzzy set μ itself and examine some properties of this topology. This is not a tolerance topology as in Chakrabarty *et al* [4] because it uses the collection of level subsets. It is also different from the topology by a collection of mere fuzzy subsets as in Chang [5]. It is also different from the topology in Shostak [23] because this topology is not on X.

Definition 1.1 ([28])**.** Let X be a non-empty set. The fuzzy subset μ of the set X is a function $\mu : X \to [0,1]$, where μ_X is the membership function of the fuzzy set μ.

We can just use μ for μ_X since it is characteristic of the fuzzy set μ

Definition 1.2 ([5, 28])**.** Let μ and λ be any two fuzzy subsets of a set X. Then

(i) λ and μ are equal if $\mu(x) = \lambda(x)$ for every x in X

(ii) λ and μ are disjoint if $\mu(x) \neq \lambda(x)$ for every x in X

(iii) $\lambda \subseteq \mu$ if $\mu(x) \geq \lambda(x)$

Definition 1.3 ([24])**.** Let μ be the fuzzy subset of X. Then, for some $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$ is called a α-level subset of the fuzzy subset μ. If $\alpha_1 \leq \alpha_2$, then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$.

Definition 1.4 ([28])**.** Let μ be a fuzzy subset of X. Then, λ is the complement of μ if $\lambda(x) = 1 - \mu(x)\ \ \forall x \in X$.

Definition 1.5 ([5])**.** The family $T = \{A_j\}_{j \in I}$ of fuzzy subsets of X such that

(i) Φ, $X \in T$,

(ii) $\cup_j A_j \in T$ for each $j \in I$ and

(iii) $A_k \cap A_i \in T$ for each $k, i \in I$

is called a *fuzzy topology* on X and (X, T) is called a *fuzzy topological space.*

Remark 1.6 ([15]). In regard to 1.5, $\Phi = \{x \in X : \mu(x) = 0 \ \ \forall x \in X\}$ and $X = \{x \in X : \mu(x) = 1 \ \ \forall x \in X\}$. Every member of T is called T-open or simply open fuzzy set. Alternative to this is the definition in [15] where κ_λ is the fuzzy set in X with constant value λ. Then, $\kappa_0 = \Phi$ and $\kappa_1 = X$.

Definition 1.7 ([5])**.** A fuzzy set U in a fuzzy topological space is a neighbourhood of a fuzzy set μ if there exists an open fuzzy set O such that $\mu \subset O \subset U$.

Remark 1.8 ([5]). The collection of all such neighbourhood U of μ is called neighbourhood system of μ. In this case, μ is called the *interior fuzzy set* of U and the collection of all the interior fuzzy sets is called the *interior* of U and can be denoted U^o. *It is important to note that a fuzzy set μ is open if and only if $\mu = \mu^o$.*

Theorem 1.9 ([5])**.** *A fuzzy set is open if and only if it is the neighbourhood of each fuzzy set in it.*

Definition 1.10 ([5])**.** A sequence of fuzzy sets, say $\{A_n : n = 1, 2, 3, ...\}$, is eventually contained in a fuzzy set A if and only if there is an integer m such that, if $n \geq m$, then $A_n \subset A$.

Definition 1.11 ([5])**.** A sequence of fuzzy sets, say $\{A_n : n = 1, 2, 3, ...\}$, is frequently contained in a fuzzy set A if and only if for each integer m there is an integer n such that, if $n \geq m$, then $A_n \subset A$.

Remark 1.12 ([5]). A sequence of fuzzy sets $\{A_n : n = 1, 2, 3, ...\}$ in a fuzzy topological space is said to converge to a fuzzy set A if and only if it is eventually contained in each neighbourhood of A.

Definition 1.13 ([5])**.** A fuzzy set A in a fuzzy topological space is a cluster fuzzy set of a sequence of fuzzy sets if and only if the sequence is frequently contained in every neighbourhood of A.

Definition 1.14 ([27])**.** A fuzzy point x_p in a space of points say, X, is a fuzzy set μ_{x_p} defined by

$$\mu_{x_p}(x) = \left\{ \begin{array}{l} \neq 0, \ \text{when } x = x_p \\ \\ 0, \ \text{elsewhere.} \end{array} \right\}$$

The implication of this is that when $\mu_{x_p}(x)$ is restricted to $X - \{x_p\}$ it is an improper (or constant) fuzzy set with membership value 0.

Theorem 1.15 ([20])**.** *Let A_α be a α-level subset of a fuzzy topological space (X, T). A point $x \in X$ is a fuzzy accumulation point or fuzzy cluster point of A_α if for T-open subset U containing x it is such that $U \subset U_\beta$ with $A_\alpha \cap U \setminus \{x\} \neq \emptyset$ and $\beta \leq \alpha$.*

Definition 1.16 ([13])**.** A class $\{A_i\}$ of sets is said to have finite intersection property if every finite subclass has a non-empty intersection.

Definition 1.17 ([13])**.** A fuzzy topological space is sequentially compact if every sequence in it converges to a point in it.

2 MAIN RESULTS

We now introduce a new topology τ^* which is defined on a fuzzy subset μ of a non-empty set X rather than on X itself. The topology is the collection of α-level subsets $\{\mu_{\alpha_i}\}$ of X.

In what follows, we shall show that the collection of α-level subsets of X defines a topology on the fuzzy subset μ of X. Some basic and useful definitions shall be given. Some of these look like Chang's definition but are modified to fit into the new topology.

Theorem 2.1. *The family $\tau^* = \{\mu_{\alpha_i} | \mu(x) \geq \alpha_i\} \setminus X$ for $\alpha_i \in [0,1]$ defines a topology on μ.*

Proof. Note that we have dropped the level subset $X = \mu_{\alpha_m}$, for some $\alpha_m = 1$ in which case $\mu(x) = 1 \ \forall x \in X$. Then, $\mu_{\alpha_t} = \{x \in X : \mu(x) \geq \alpha_t \in [0,1]\} = \mu$ and $\mu_{\alpha_n} = \{x \in X : \mu(x) = 0 \ \forall x \in X \text{ for some } \alpha_n = 0\} = \mu_\emptyset$. Hence,

$$\mu, \mu_\emptyset \in \tau^*.$$

If we also consider that $\mu_{\alpha_1}, \mu_{\alpha_2} \in \tau^*$ and take $\beta = max\{\alpha_1, \alpha_2\}$. By 1.3,

$$\mu_{\alpha_1} \cap \mu_{\alpha_2} = \mu_\beta \in \tau^*.$$

Further, for $\{\mu_{\alpha_i}\}_{i=1}^{\infty}$ such that for every i, $\mu_{\alpha_i} \in \tau^*$. Let $\gamma = min\{\alpha_i\}$. By 1.3,

$$\cup_{i=1}^{\infty} \mu_{\alpha_i} = \mu_\gamma \in \tau^*. \quad \square$$

Remark 2.2. It should be noted that this family has a noetherian property. In this regard, (μ, τ^*) is a topological space and this we refer to as fuzzy level topological space or τ^*-space. Every element $\mu_\alpha \in \tau^*$ is called level open or τ^*-open. Hence, a level set μ_β is τ^*-closed if and only if its complement μ'_β is τ^*-open. This level openness agrees to the definition of level openness by [2], [12] and [23]. A fuzzy set λ can be said to be open if it coincides with a level subset of μ. When $\tau^* = \{\mu, \mu_\emptyset\}$ we have an indiscrete level topological space. If $\tau^* = \{\mu, \mu_\emptyset, \mu_{\alpha_1}, \mu_{\alpha_2}, \mu_{\alpha_3}, ..., \mu_{\alpha_n}\}$, where $\{\mu_{\alpha_i}\}$ is a collection of all possible level subsets of μ, we have a discrete level topological space.

Definition 2.3. The sequence of level subsets $\{\mu_{\alpha_i}\}$ in τ^* is frequently contained in a level subset μ_{α_k} if for each i there is an i_0 such that for $\alpha_{i_0} \geq \alpha_i$ we have $\mu_{\alpha_i} \subseteq \mu_{\alpha_k}$.

Definition 2.4. The sequence of level subsets $\{\mu_{\alpha_i}\}$ is eventually contained in a level subset μ_{α_k} if there is an i_0 such that for $\alpha_i \geq \alpha_{i_0}$ we have $\mu_{\alpha_i} \subseteq \mu_{\alpha_k}$.

Definition 2.5. The sequence of level subsets $\{\mu_{\alpha_i}\}$ is said to converge to a level subset μ_m if the sequence is eventually contained in every neighbourhood of μ_m.

Definition 2.6. A level subset μ_{α_0} in the sequence $\{\mu_{\alpha_i}\}$ is called maximal if $\mu_{\alpha_i} \subseteq \mu_{\alpha_0}$ for all i such that $\alpha_0 \leq \alpha_i$ or $\alpha_0 = min\{\alpha_i\}$ and it is minimal if $\mu_{\alpha_0} \subseteq \mu_{\alpha_i}$ for all i such that $\alpha_0 \geq \alpha_i$ or $\alpha_0 = max\{\alpha_i\}$.

Remark 2.7. It is can be observed that the sequence is frequently contained in the maximal level subset. This is because, for any α_i there is an α_k such that $\alpha_i \geq \alpha_k \geq \alpha_0$, then $\mu_{\alpha_i} \subseteq \mu_{\alpha_0}$.

Definition 2.8. The sequence $\{\mu_{\alpha_i}\}$ of fuzzy level subsets is bounded if it has both the minimal and maximal level subsets.

Definition 2.9. A fuzzy level subset μ_{α_0} in (μ, τ^*) is a cluster fuzzy level subset of a sequence of fuzzy level subset $\{\mu_{\alpha_i}\}$ if the sequence is frequently contained in every neighbourhood of μ_{α_0}.

Definition 2.10. A fuzzy subset λ in (μ, τ^*) is a neighbourhood of a level subset μ_{α_k} if there is a level open set O_{α_m} such that $\mu_{\alpha_k} \subset O_{\alpha_m} \subset \lambda$.

Remark 2.11. Though it cannot be said that an element x is in μ, except only to a degree, but rather it can be said that $x \in \mu_\alpha$. Hence, we can define a fuzzy subset μ as a neighbourhood of an element x if there is a τ^*-open level subset O_α such that $x \in O_\alpha \subset \mu$. Definition 2.10 is a generalisation of this since if μ is a neighbourhood of any μ_α, it is a neighbourhood of each point of μ_α.

Definition 2.12. The level subset μ_{α_k} in 2.10 is called an interior fuzzy level subset of the fuzzy set λ and each point of μ_{α_k} is called fuzzy interior point of λ. The collection of all such μ_{α_k} is the interior of λ.

Definition 2.13. The collection of all such neighbourhoods λ of level subset (or of points of) μ_{α_k} in 2.10 is called the neighbourhood system (or of the points) of μ_{α_k} and is denoted N.

Theorem 2.14. *A fuzzy subset λ in a fuzzy topological space (μ, τ^*) is τ^*-open if and only if λ is a neighbourhood of every μ_{α_i} in it.*

Proof. Let λ be open in μ and $\mu_{\alpha_i} \subset \lambda$ for each i. Then, λ being opened is in τ^*. By 2.2, there is a $\beta \in I\!R$ such that $\mu_{\alpha_i} \subset \lambda = \mu_\beta$ with $\beta < \alpha_i$. Since β and α_i are distinct real numbers, (β, α_i) forms an interval. So there is a $\gamma \in I\!R$ such that $\beta < \gamma < \alpha_i$ which implies that $\mu_{\alpha_i} \subset \mu_\gamma \subset \mu_\beta$. But $\mu_\gamma \in \tau^*$. Hence, for every $\mu_{\alpha_i} \subset \lambda$, there is $\mu_\gamma \in \tau^*$ such that

$$\mu_{\alpha_i} \subset \mu_\gamma \subset \lambda = \mu_\beta.$$

Thus, λ is a neighbourhood of all μ_{α_i}'s in it.

Conversely, let λ be a neighbourhood of each μ_{α_i} in it. Then, $\cup\mu_{\alpha_i} \subseteq \lambda$. But each $\mu_{\alpha_i} \subset \lambda$ for each i so that for each $x \in \mu_{\alpha_i} \subset \lambda$, $\mu(x) \geq \alpha_i$. But each x which is to a certain degree contained in λ is in some μ_{α_i} so that $\lambda \subseteq \cup\mu_{\alpha_i}$. Then,

$$\lambda = \cup\mu_{\alpha_i} \in \tau^*.$$

Thus, λ is open. □

The following theorem connects Chang's topological space with τ^*-space.

Theorem 2.15. *Let the fuzzy set λ be a neighbourhood of the fuzzy set μ in Chang's topological space. Then, λ is a neighbourhood of all the τ^*-open subsets μ_α of μ.*

Proof. Note that $\mu_\alpha \subset \mu$ for every α. Since λ is a neighbourhood of μ, by 1.7 there is an open set O in Chang's topological space such that

$$\mu \subset O_\alpha \subset \lambda.$$

But for every α, $\mu_\alpha \subset \mu \subset O \subset \lambda$. Hence, $\mu_\alpha \subset O \subset \lambda$. □

Theorem 2.16. *A sequence of fuzzy subsets that is frequently contained in a fuzzy subset μ is eventually contained in μ.*

Proof. By 1.11 for each m such that $n \geq m$, we have $A_n \subset A$. But we can choose a fix m_0 such that for $n \geq m_0$, $A_n \subset A$. Hence, by 1.10 $\{A_n\}$ is eventually contained in A. □

Theorem 2.17. *Every sequence of level subsets $\{\mu_{\alpha_i}\}$ of fuzzy set μ is frequently contained in the fuzzy set μ.*

Proof. For each m such that $\alpha_i \geq m$, $\mu_{\alpha_i} \subseteq \mu_m$. Since for each m, $\mu_{\alpha_i} \subseteq \mu_m \subset \mu$, we have $\mu_{\alpha_i} \subseteq \mu$. By 1.11, the sequence $\{\mu_{\alpha_i}\}$ is frequently contained in μ. □

Remark 2.18. By 2.16, the sequence $\{\mu_{\alpha_i}\}$ is eventually contained in μ.

Theorem 2.19. *Every sequence of level subsets $\{\mu_{\alpha_i}\}$ of fuzzy set μ converges to a unique μ_{α_0}, where $\alpha_0 = max\{\alpha_i\}$ or μ_{α_0} is the minimal fuzzy level subset in the sequence.*

Proof. Let μ_α be any neighbourhood of μ_{α_0}. Then there is a τ^*-open set μ_{α_k} such that $\mu_{\alpha_0} \subset \mu_{\alpha_k} \subset \mu_\alpha$ with $\alpha_0 > \alpha_k > \alpha$. All such μ_{α_k} are in the sequence $\{\mu_{\alpha_i}\}$. For a neighbourhood system of μ_{α_0} we can have a collection of such α_k's. Let $\alpha_m = min\{\alpha_k\}$ so that we have an α_m such that with $\alpha_k \geq \alpha_m$, $\mu_{\alpha_k} \subseteq \mu_{\alpha_m} \subset \mu_\alpha$. This implies that in the sequence $\{\mu_{\alpha_i}\}$, there is an α_m such that if $\alpha_i \geq \alpha_m$, $\mu_{\alpha_i} \subset \mu_\alpha$. The sequence $\{\mu_{\alpha_i}\}$ is arbitrary and so is the neighbourhood μ_α of μ_{α_0}. By 2.4, any sequence $\{\mu_{\alpha_i}\}$ is eventually contained in every neighbourhood μ_α of μ_{α_0}. By 2.5, the sequence converges to μ_{α_0}. For any α_i, since, $\{\mu_{\alpha_i}\}$ is a sequence of fuzzy level subsets, by convergence, $\mu_{\alpha_0} \subseteq \mu_{\alpha_i}$ for which $\alpha_0 \geq \alpha_i$.

We now show that μ_{α_0} is unique and minimal in $\{\mu_{\alpha_i}\}$. For any α_i, since, $\{\mu_{\alpha_i}\}$ is a sequence of fuzzy level subsets, by convergence, $\mu_{\alpha_0} \subseteq \mu_{\alpha_i}$ for which $\alpha_0 \geq \alpha_i$. Assume there is another α_j such that $\alpha_j \geq \alpha_i$ for all i but $\alpha_0 \neq \alpha_j$. Then there is a μ_{α_j} such that $\alpha_j > \alpha_0$, $\mu_{\alpha_j} \subset \mu_{\alpha_0}$. But $\mu_{\alpha_j} \in \{\mu_{\alpha_i}\}$ which converges to μ_{α_0} so that $\mu_{\alpha_0} \subset \mu_{\alpha_j}$, with $\alpha_j < \alpha_0$. This is not only a contradiction but also shows that $\mu_{\alpha_0} = \mu_{\alpha_i}$ and that $\alpha_0 = \alpha_j$. □

Theorem 2.20. *Every level subset of μ in the sequence $\{\mu_{\alpha_i}\}$, except the maximal one, is a fuzzy cluster level subset.*

Proof. Let μ_{α_0} be any fuzzy level subset of μ and μ_α the neighbourhood of μ_{α_0}. Always, $\mu_{\alpha_0} \subset \mu_\alpha$ with $\alpha_0 > \alpha$. For each α such that $\alpha_i \geq \alpha$, $\mu_{\alpha_i} \subseteq \mu_\alpha$ and so by 2.9 the sequence is frequently contained in μ_α. As for the maximal one, it is not properly contained in any other level subset which can serve as its neighbourhood. □

Corollary 2.21. *The sequence $\{\mu_{\alpha_i}\}$ converges to its minimal fuzzy level cluster subset μ_{α_0}.*

Proof. The sequence converges to its minimal level subset by 2.19. And by 2.20 it is one of the fuzzy cluster level subset. □

Theorem 2.22. *The fuzzy set μ is open if and only if it can be expressed as the union of all its fuzzy cluster level subsets.*

Proof. If $\mu = \cup_{\alpha_i}\mu_{\alpha_i} = max\{\mu_{\alpha_i}\}$. Then there is an $\alpha_k = min\{\alpha_i\}$ such that $\mu = \mu_{\alpha_k} \in \tau^*$. Thus, μ is open.

Conversely, if μ is open, we have $\mu \in \tau^*$. Then, there is an α_k such that $\mu_{\alpha_k} = \mu$. □

Remark 2.23. By 2.10 and 2.12, the level subsets μ_α's are the interiors of μ and 2.22 is like saying that a set is open if and only if it contains all its interior points. Also, Definition 2.9 guarantees that for every neighbourhood μ_α of μ_{α_0} we can have an open set $O_{\alpha_k} \in \tau^*$ such that $\mu_{\alpha_0} \subset O_{\alpha_k} \subset \mu_\alpha$ so that for all $x \in \mu_{\alpha_0}$ we have some $y \in O_{\alpha_k}$ such that $x \neq y$. Then for all such open sets O_{α_k}, it is guaranteed that

$$\mu_\alpha \cap O_{\alpha_k} \setminus \mu_{\alpha_0} \neq \emptyset.$$

Furthermore, this also guarantees that every open set which contains μ_{α_0} contains other points which are not in μ_{α_0} but to a certain degree are contained in μ. With this construction, it is also possible to define the fuzzy cluster point of a level set μ_{α_i} for any i which will coincide with the definition of the same as given by Onasanya in [20]. In such case we say that an x which to a certain degree belongs to a fuzzy set μ is a fuzzy cluster point of a level subset μ_{α_i} of a fuzzy set μ if for every τ^*-open set O_{α_k} containing x,

$$\mu_{\alpha_i} \cap O_{\alpha_k} \setminus \{x\} \neq \emptyset.$$

Then, U and U_β as defined by Onasanya in [20] can be taken respectively as O_{α_k} and μ_α in which case $A_\alpha \cap U \setminus \{x\} \neq \emptyset$ with $\beta \leq \alpha$ is like $\mu_{\alpha_i} \cap O_{\alpha_k} \setminus \{x\} \neq \emptyset$ for $O_{\alpha_k} \subseteq \mu_\alpha$ with $\alpha \leq \alpha_k$.

Theorem 2.24. *Every sequence of level subsets $\{\mu_{\alpha_i}\}$ is bounded.*

Proof. By 2.19, the sequence has a minimal level subset μ_{α_0} so that $\mu_{\alpha_0} \subseteq \mu_{\alpha_i}$ with $\alpha_i \leq \alpha_0$ for all i. We now show that it also has a maximal level subset. Note that, for every i, $\mu_{\alpha_i} \subseteq \mu$. By 2.17, we can have some m such that for every m $\alpha_i \geq \alpha_m$ and $\mu_{\alpha_i} \subseteq \mu_{\alpha_m} \subseteq \mu$. If we fix the minimum of such α_m to be $\alpha_{m'}$, it can be shown that $\mu_{\alpha_{m'}} = max\{\mu_{\alpha_m}\} = max\{\mu_{\alpha_i}\}$. Since $\alpha_{m'} = min\{\alpha_m\}$, $\alpha_{m'} \leq \alpha_m$ and that implies that $\mu_{\alpha_m} \subseteq \mu_{\alpha_{m'}}$ for all m. But note that $\alpha_i \geq \alpha_m \geq \alpha_{m'}$. Then, $\mu_{\alpha_i} \subseteq \mu_{\alpha_m} \subseteq \mu_{\alpha_{m'}}$ for all i. By 2.6, $\mu_{\alpha_{m'}} = max\{\mu_{\alpha_m}\} = max\{\mu_{\alpha_i}\}$. Then the sequence also has a maximal level subset. Combining the minimality of μ_{α_0} and the maximality of $\mu_{\alpha_{m'}}$,

$$\mu_{\alpha_0} \subseteq \mu_{\alpha_i} \subseteq \mu_{\alpha_{m'}}.$$

By 2.8, $\{\mu_{\alpha_i}\}$ has both the maximal and minimal level subset and so is bounded. □

Remark 2.25. The following is similar to what we have in classical case that a set is closed if and only if it contains all its limit points, in which case the set is bounded.

Theorem 2.26. *The sequence of level subsets is bounded if and only if it contains all its cluster level subsets of the fuzzy set.*

Proof. If a sequence contains all the cluster level subsets of the fuzzy set, they are also level subsets of that fuzzy set by 2.20. By 2.24, the sequence is bounded.

Conversely, if the sequence is bounded, it has both the minimal and maximal level subsets by 2.8. Since it is a noetherian sequence, it contains the maximal level subset, and thus the sequence, contains all the other level subsets of the fuzzy set. But all of them are cluster level subsets,except the maximal one, by 2.20. □

Definition 2.27 ([13])**.** The sequence of level subsets $\{\mu_{\alpha_i}\}$ is a cover for μ_{α_0} if $\mu_{\alpha_0} \subset \cup\mu_{\alpha_i}$.

Remark 2.28. The property of compactness of μ is hard to come by in this space because of the neotherian property of $\{\mu_{\alpha_i}\}$. We rather can have something that mimics it. As a matter of fact, for the same reason, this space does not separate points. So, the property of Hausdorffness is not possible. But the space (μ, τ^*) has something that much resembles sequential compactness [3].

Definition 2.29. The sequence of level subsets $\{\mu_{\alpha_i}\}$ is a quasicover for μ_{α_0} if $\mu_{\alpha_0} \subseteq \cup\mu_{\alpha_i}$.

Definition 2.30. A level subset μ_{α_0} is quasicompact if every open quasicover $\{\mu_{\alpha_i}\}$ has a refinement or a subsequence $\{\mu_{\alpha_j}\}$ with $\alpha_j \geq \alpha_i$ which is a quasicover of μ_{α_0}.

Theorem 2.31. *Every cluster level subset μ_{α_0} of μ in a τ^*-space, apart from the maximal one, is quasicompact.*

Proof. Let $\{\mu_{\alpha_i}\}$ be the sequence which is a quasicover for μ_{α_0}. Then, $\mu_{\alpha_0} \subseteq \cup\mu_{\alpha_i}$. By 2.6 any such level subset μ_{α_0} of μ is not maximal and by 2.20 is a cluster level subset. Let μ_{α_k} be any neighbourhood of μ_{α_0}, then, $\mu_{\alpha_0} \subset \mu_{\alpha_k}$ so that $\alpha_0 > \alpha_k$. By 2.3 and 2.9 and the fact that μ_{α_k} is not minimal in $\{\mu_{\alpha_i}\}$, we have some $\alpha_j \in \{\alpha_i\}$ such that and $\alpha_j \geq \alpha_k$ and $\mu_{\alpha_j} \subseteq \mu_{\alpha_k}$. Hence, $\alpha_0 \in \{\alpha_j\}$ and $\mu_{\alpha_0} \in \{\mu_{\alpha_j}\}$ so that $\mu_{\alpha_j} \subseteq \mu_{\alpha_k}$, $\mu_{\alpha_0} \subseteq \cup\mu_{\alpha_j}$. Since $\cup\mu_{\alpha_j}$ is for some $\alpha_j \in \{\alpha_i\}$ there are still some α_m's in $\{\alpha_i\}$ such that $\alpha_m < \alpha_k \leq \alpha_j$ and we have $\cup\mu_{\alpha_j} \subset \cup\mu_{\alpha_i}$. Thus, we have some μ_{α_m}'s which are in $\{\mu_{\alpha_i}\}$ but are not in $\{\mu_{\alpha_j}\}$ in which case, $\{\mu_{\alpha_j}\}$ is a subsequence of $\{\mu_{\alpha_i}\}$. □

Definition 2.32. A fuzzy topological space is parasequentially compact if every sequence of level subsets in it converges to a level subset in it.

Theorem 2.33. *(μ, τ^*) is a parasequentially compact space.*

Proof. By 2.19, every sequence of level subsets in (μ, τ^*) converges to a level subset $\mu_{\alpha_0} \subseteq \mu$. Hence, by 2.32, (μ, τ^*) is parasequentially compact. □

Theorem 2.34. *The sequence of level subsets $\{\mu_{\alpha_i}\}$ of (μ, τ^*) has finite intersection property.*

Proof. Let $\{\mu_{\alpha_i}\}$ be any sequence in (μ, τ^*). If α_i and α_{i+1} are such that $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_i \leq \alpha_{i+1} \cdots$, then $\cdots \mu_{\alpha_{i+1}} \subseteq \mu_{\alpha_i} \subseteq \cdots \subseteq \mu_{\alpha_1} \subseteq \mu_{\alpha_0}$. The sequence is noetherian. The maximal one contains any other and the minimal one is contained in all others. Hence, they all have at least the minimal one, say μ_{α_0}. Therefore, any finite subsequence $\{\mu_{\alpha_j}\}_{j=1}^{n}$ of $\{\mu_{\alpha_i}\}$ is such that

$$\cap \mu_{\alpha_j} \supseteq \mu_{\alpha_0} \neq \emptyset.$$

Thus,

$$\cap \mu_{\alpha_j} \neq \emptyset.$$

By 1.16, the sequence $\{\mu_{\alpha_i}\}$ has a finite intersection property. □

Theorem 2.35 ([18])**.** *Let μ be a fuzzy normal subgroup of a group G. Let any $\alpha_i \in [0, 1]$ such that $\alpha_i \leq \mu(e)$, where e is the identity of G. Then μ_{α_i} is a normal subgroup of G.*

Remark 2.36. It has been shown in [18] that the collection $\{\mu_{\alpha_i}\}$ as defined in 2.35 form a chain of normal subgroups of G.

Example 2.37. Let $T = \{G_i\}_{i=0}^{n}$ be a collection of normal subgroups of G which form an invariant series. Then, $T^* = T \cup \emptyset = \{G_i\}_{i \in I}$, such that $\emptyset \subseteq G_i \ \ \forall i \in I$, is a topology on G so that (G, T^*) is a topological space since

a. $\emptyset, G \in T^*$

b. $G_n \cap G_k = G_l \in T^*$ where $l = n$ (if $G_n \subseteq G_k$) or $l = k$ (if $G_k \subseteq G_n$) for any $k, n \in I$

c. $\cup_k G_k = G_m \in T^*$ where G_m is the largest of G_k's.

Then, all the normal subgroups $G_j \in T$ such that $G_j \neq G$ are quasicompact.

Example 2.38. Let G be a group and μ a fuzzy normal subgroup of G. Then, the collection of α_i-level subgroups $\{\mu_{\alpha_i}\}$ of μ form a chain of normal subgroup of G by 2.35 and 2.36. If we exempt μ_{α_i} for which $\alpha_i = 1$, which is G, by 2.1, $\tau^* = \{\mu_{\alpha_i}\} \setminus G$ is a topology on μ. Every element of τ^* except the maximal one is quasicompact.

In what follows, we give some algebraic properties of τ^*. If we take the subset of the family $\tau^* = \{\mu_{\alpha_i}\}$ to be the subfamily $\{\mu_{\alpha_k}\}$ such that $\alpha_k \geq \alpha_i$ for each k and i, then the following results are very straightforward to prove though it is not out of place to point them out.

Lemma 2.39. *The relation $\sim$ on τ^* defined by $\mu_{\alpha_k} \sim \mu_{\alpha_m}$ if either $\mu_{\alpha_k} \subseteq \mu_{\alpha_m}$ (that is $\alpha_m \geq \alpha_k$) or $\mu_{\alpha_m} \subseteq \mu_{\alpha_k}$ (that is $\alpha_k \geq \alpha_m$) is such that $(\tau^*, \sim)$ is well ordered.*

Proof. This is straightforward. □

Lemma 2.40. *Let $\{\alpha_i\}$ be the level set (or set of levels) and $\{\mu_{\alpha_i}\}$ the α_i-fuzzy level subsets of μ. Then, $\alpha_k \sim \alpha_m \Longleftrightarrow \mu_{\alpha_m} \sim \mu_{\alpha_k}$.*

Proof. This is straightforward. □

CONCLUSION

The theory of fuzzy sets has advanced in a variety of ways and in many disciplines. As mentioned in [30] an also the beginning of this paper, applications of this theory can be found, for example, in artificial intelligence, computer science, control engineering,decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Theoretical advances have been made in many directions.

We have shown that it is possible to define a topology on a set X in terms of the α-level subsets of its fuzzy subsets. Such topology, however, does not give room for compactness but rather a property that mimics it.

Some application can be found e.g. in [8, 14, 16, 17, 21, 29] and many others.

CONFLICT OF INTERESTS

The authors declare that they have no conflict of interests.

Acknowledgments

The work presented in this paper was supported within the project for "Development of basic and applied research developed in the long term by the departments of theoretical and applied bases of FMT (Project code: DZRO K-217) supported by the Ministry of Defence of the Czech Republic.

References

[1] M. Al Tahan, S, Hoskova-Mayerova, B. Davvaz, An overview of topological hypergroupoids, J. Intell. Fuzzy Systems, (2018), 34(3), 1907-1916, doi. 10.3233/JIFS-171265.

[2] S. S. Benchalli and G. P. Siddapur, On the level spaces of fuzzy topological spaces, Bull. of Math. Anal. Appl. 1(2) (2009), 57–65.

[3] H. Bordbar, I. Cristea, M. Novak, Height of Hyperideals in Noetherian Krasner Hyperrings, University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, 79(2), 31-42, (2017).

[4] M. K. Chakrabarty and T.M.G. Ahsanullah, Fuzzy topology on fuzzy sets and tolerance topology, Fuzzy Sets and Systems 45 (1992), 103–108.

[5] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. and Appl. 24 (1968), 182–190.

[6] A. K. Chaudhuri and P. Das, Some results on fuzzy topology on fuzzy sets, Fuzzy Sets and Systems 56 (1993), 331–336.

[7] I. Chon, Some properties of fuzzy topological groups, Fuzzy Sets and Systems 123 (2001), 197–201.

[8] I. Cristea, S. Hoskova, Fuzzy topological hypergroupoids, Iran. J. Fuzzy Syst., 6(4) (2009), 1321.

[9] I. Cristea, J. M. Zhan, Lower and upper fuzzy topological subhypergroups, Acta Math. Sin. (Engl. Ser.), 29 (2), 315-330, doi:10.1007/s10114-012-1170-4, (2013).

[10] P. Das, Fuzzy topology on fuzzy sets: product fuzzy topology and fuzzy topological groups, Fuzzy Sets and System 109 (1998), 367–372.

[11] T. E. Gantner, Compactness in fuzzy topological spaces, J. Math. Anal. Appl. 62 (1978), 547–562.

[12] V. Gregori and A. Vidal, Fuzziness in Chang's fuzzy topological spaces, Rend. Istit. Mat. Univ. Trieste XXX (1999), 111–121.

[13] S. Lipschutz, Theory and problems of general topology, Schaum's Outline Series, McGraw-Hill Book Company, New York, 1965.

[14] S. Hoskova, Topological hypergroupoids, Computers and Mathematics with Applications, 64(9), 2845–2849.

[15] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976), 621–633.

[16] A. Maturo, Fuzzy Measures and Coherent Join Measures, International Journal of Intelligent Systems, 26 (12), (2011), 1196-1205, doi: 10.1002/int.20512.

[17] F. Maturo, V. Ventre, Consensus in Multiperson Decision Making Using Fuzzy Coalitions. Studies in Fuzziness and Soft Computing, (2017), 451464. doi:10.1007/978-3-319-60207-$3_2$6.

[18] N. P. Mukherjee and P. Bhattacharya, Fuzzy normal subgroups and fuzzy cosets, Information Sciences. 34 (1984), 225–239.

[19] M. Norouzi, I. Cristea, A new type of fuzzy subsemihypermodules, J. Intell. Fuzzy Systems, 32(3), 1711-1717, doi: 10.3233/JIFS-151867, (2017).

[20] B. O. Onasanya, Some topological concepts via level subsets of a fuzzy subset, Advances in Fuzzy Sets and Systems 20(2) (2015), 177–188.

[21] A. Saeid, C. Flaut, S. Hoskova-Mayerova, et al., Some connections between BCK-algebras and n-ary block codes, Soft Comput (2018) 22(1), 41–46. doi:10.1007/s00500-017-2788-z.

[22] M. Sarkar, On fuzzy topological spaces, J. Math. Anal. and Appl. 79 (1981), 384–394.

[23] A. P. Shostak, On a fuzzy topological structure, Rend. Istit. Circ. Mat. Palermo Ser II(11) (1985), 89–103.

[24] W.B. Vasantha Kandasamy, Smarandache fuzzy algebra, American Research Press, Rehoboth (2003).

[25] C. K. Wong, Covering Properties of fuzzy topological spaces, J. Math. Anal. and Appl. 43 (1973), 697–704.

[26] C. K. Wong, Fuzzy topology: product and quotient theorems, J. Math. Anal. and Appl. 45 (1974), 512–521.

[27] C. K. Wong, Fuzzy points and local properties of fuzzy topology, J. Math. Anal. and Appl. 46 (1974), 316–328.

[28] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965), 338–353.

[29] L. Žák, D., Vališ, Comparison of regression and fuzzy estimates on field oil data, Mendel (2015), 85–902, Code 113282.

[30] H. J. Zimmermann, Fuzzy Set Theory-and Its Applications, Fourth edition, Springer Science Business Media, New York (2001).

Babatunde Oluwaseun Onasanya,
Department of Mathematics, Faculty of Science,
University of Ibadan,
Ibadan, Nigeria.
Email: babtu2001@yahoo.com

Sarka Hoskova-Mayerova,
Department of Mathematics and Physics, Faculty of Military Technology,
University of Defence,
Brno, Czech Republic.
Email: sarka.mayerova@unob.cz